Problem Set 4 – Relativity (`G0I36A`)

1 Coordinate Transformations: Minkowski in Disguise!

1.) First, we note that the chain rule can be used easily to express the differentials dx^μ in terms of the differentials $dx^{\mu'}$ of the new coordinates:

$$dx^\rho = \frac{\partial x^\rho}{\partial x^{\mu'}} dx^{\mu'} \ . \tag{1.1}$$

Note that the index μ is a dummy index that we sum over, so this equation secretly has many terms in it, one for each dimension. With this in mind, we can express the metric as follows:

$$\begin{aligned} ds^2 &= g_{\rho\sigma} dx^\rho dx^\sigma \\ &= g_{\rho\sigma} \left(\frac{\partial x^\rho}{\partial x^{\mu'}} dx^{\mu'} \right) \left(\frac{\partial x^\sigma}{\partial x^{\nu'}} dx^{\nu'} \right) \\ &= \left(\frac{\partial x^\rho}{\partial x^{\mu'}} \frac{\partial x^\sigma}{\partial x^{\nu'}} g_{\rho\sigma} \right) dx^{\mu'} dx^{\nu'} \ . \end{aligned} \tag{1.2}$$

Note that there's nothing special about the indices we chose; they are dummy indices that we sum over, so we can call them whatever we want. So, if you're confused about why my indices might look different than the ones you chose, just remember that it doesn't matter for dummy indices.

Now, the distance ds between two points infinitesimally close to one another is a scalar quantity, and so it should be invariant under coordinate transformations. At the same time, though, in the new coordinate frame $x^{\mu'}$ there should be some new metric $g_{\mu'\nu'}$ that we use to measure distances between points, i.e.

$$ds^2 = g_{\mu'\nu'} dx^{\mu'} dx^{\nu'} \ . \tag{1.3}$$

If we compare (1.2) and (1.3), we find that the only way to make the two match is to set

$$\boxed{g_{\mu'\nu'} = \frac{\partial x^\rho}{\partial x^{\mu'}} \frac{\partial x^\sigma}{\partial x^{\nu'}} g_{\rho\sigma}} \ , \tag{1.4}$$

just as we set out to prove.

2.) From the given coordinates, you should easily be able to see that

$$\frac{dt'}{dt} = \alpha_0 \ , \quad \frac{dx'}{dx} = \alpha_1 \ , \quad \frac{dy'}{dy} = \alpha_2 \ , \quad \frac{dz'}{dz} = \alpha_3 \ . \tag{1.5}$$

Another way of writing this is

$$ds^2 = -\frac{c^2}{\alpha_0^2}(dt')^2 + \frac{1}{\alpha_1^2}(dx')^2 + \frac{1}{\alpha_2^2}(dy')^2 + \frac{1}{\alpha_3^2}(dz')^2 \ . \tag{1.6}$$

We can also write this as $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, where the metric tensor $g_{\mu\nu}$ in the new coordinates is given by

$$g_{\mu\nu} = \begin{pmatrix} -\frac{c^2}{\alpha_0^2} & 0 & 0 & 0 \\ 0 & \frac{1}{\alpha_1^2} & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha_2^2} & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha_3^2} \end{pmatrix} . \tag{1.7}$$

The inverse metric should be pretty easy to see for such a simple diagonal metric:

$$g^{\mu\nu} = \begin{pmatrix} -\frac{\alpha_0^2}{c^2} & 0 & 0 & 0 \\ 0 & \alpha_1^2 & 0 & 0 \\ 0 & 0 & \alpha_2^2 & 0 \\ 0 & 0 & 0 & \alpha_3^2 \end{pmatrix} . \tag{1.8}$$

You should also note that $\beta_{0,1,2,3}$ do not appear in any of these expressions. This is because the β's are simply shifts in the coordinates. Such constant shifts cannot affect the distance between two points, though, since both points will be shifted by the same amount. Therefore the metric should be independent of β, which is what we find.

3.) We want to find a coordinate transformation, i.e. some expressions $t(u, v)$ and $x(u, v)$, such that the two metrics are equal:

$$ds^2 = dv^2 - v^2du^2 = -c^2dt^2 + dx^2 . \tag{1.9}$$

By the chain rule, we can express dt and dx as

$$dt = \frac{\partial t}{\partial u}du + \frac{\partial t}{\partial v}dv , \quad dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv . \tag{1.10}$$

With this, the Minkowski metric can then be written as

$$\begin{aligned} ds^2 &= -c^2\left(\frac{\partial t}{\partial u}du + \frac{\partial t}{\partial v}dv\right)^2 + \left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right)^2 \\ &= \left[-c^2\left(\frac{\partial t}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial u}\right)^2\right]du^2 + \left[-c^2\left(\frac{\partial t}{\partial v}\right)^2 + \left(\frac{\partial x}{\partial v}\right)^2\right]dv^2 \\ &\quad + 2\left[-c^2\frac{\partial t}{\partial u}\frac{\partial t}{\partial v} + \frac{\partial x}{\partial u}\frac{\partial x}{\partial v}\right]du\,dv . \end{aligned} \tag{1.11}$$

If we want this to match with $ds^2 = dv^2 - v^2du^2$, we need the coefficients of du^2, dv^2, and $dudv$ to match. This gives us three sets of equations, listed respectively:

$$\begin{aligned} -c^2\left(\frac{\partial t}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial u}\right)^2 &= -v^2 , \\ -c^2\left(\frac{\partial t}{\partial v}\right)^2 + \left(\frac{\partial x}{\partial v}\right)^2 &= 1 , \\ -c^2\frac{\partial t}{\partial u}\frac{\partial t}{\partial v} + \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} &= 0 . \end{aligned} \tag{1.12}$$

If you got to this point but are stuck on how to actually solve these equations, that's okay. This is a class on relativity, not partial differential equations, so I understand if this looks a little intimidating. It's actually not that bad, though, and we'll go through now how to actually break it down to find solutions.

We first consider the first equation. We need both terms in the left-hand side of the equation to have a factor of v^2 in them in order to match the right-hand side. But, since the derivatives in this equation are with respect to u and not v, this means that the simplest way to get this overall v^2 dependence is to make an ansatz of the form

$$t(u,v) = \frac{1}{c} v f(u) \ , \quad x(u,v) = v h(u) \ , \tag{1.13}$$

for some functions f and h. (We chose the factor of $1/c$ in our ansatz for t in order to get rid of the c^2 factor in the equation.) The three equations (1.12) on this ansatz (1.13) become the following three equations:

$$\begin{aligned} -f'(u)^2 + h'(u)^2 &= -1 \ , \\ -f(u)^2 + h(u)^2 &= 1 \ , \\ -f(u)f'(u) + h(u)h'(u) &= 0 \ . \end{aligned} \tag{1.14}$$

The second equation should immediately catch your eye, because it can be used to solve for one of the two functions. If we solve for h in terms of f, we find that

$$h(u) = \sqrt{1 + f(u)^2} \ . \tag{1.15}$$

From here, we can plug this into the remaining two equations (i.e. the first and the third in (1.14)) to find differential equations for f. In fact, the third equation ends up being zero trivially, so only the first equation is relevant. It is now given by

$$-\frac{f'(u)^2}{1 + f(u)^2} = -1 \ , \tag{1.16}$$

or equivalently we can write this as

$$f'(u)^2 = 1 + f(u)^2 \ . \tag{1.17}$$

This relation I hope looks at least a little familiar; it is a differential equation definition of the function $\sinh(u)$, since the derivative of $\sinh(u)$ is $\cosh(u)$, and these functions satisfy $\cosh(u)^2 = 1 + \sinh(u)^2$. Thus, we have shown that $f(u) = \sinh u$, which in turn means that (using (1.15)) we also know that $h(u) = \cosh u$. Finally, plugging f and h back into our ansatz (1.13), we find the following final result:

$$\boxed{t(u,v) = \frac{1}{c} v \sinh u \ , \quad x(u,v) = v \cosh u} \ . \tag{1.18}$$

That is, this is precisely the coordinate transformation needed to go between the two metrics given in this problem.

4.) The first thing you should notice in this problem is that the quantities $\sqrt{x^2 + y^2}$ and $\arctan(y/x)$ show up a lot. These should be familiar; they are the radial coordinate r and the angular coordinate ϕ in cylindrical coordinates, as we looked at in question 11 in problem set 3. The metric in cylindrical coordinates is

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\phi^2 + dz^2 \ . \tag{1.19}$$

The relationship between the coordinates (t', x', y', z') and these cylindrical coordinates (t, r, ϕ, z) is as follows:

$$\begin{aligned} t' &= t \\ x' &= r \cos(\phi - \omega t) \\ y' &= r \sin(\phi - \omega t) \\ z' &= z \end{aligned} \tag{1.20}$$

Equivalently, we can easily invert these relationships and write the cylindrical coordinates in terms of the new ones:

$$\begin{aligned} t &= t' \\ r &= \sqrt{x'^2 + y'^2} \\ \phi &= \arctan(y'/x') + \omega t \\ z &= z' \end{aligned} \tag{1.21}$$

Now that we have these, we need to compute the components of the metric tensor $g_{\mu'\nu'}$ for the new coordinates. The master equation that will guide us is

$$g_{\mu'\nu'} = \frac{\partial x^\rho}{\partial x^{\mu'}} \frac{\partial x^\sigma}{\partial x^{\nu'}} g_{\rho\sigma} \ , \tag{1.22}$$

where $x^{\mu'}$ and $x^{\nu'}$ refer to the new coordinates and x^ρ and x^σ refer to the old coordinates.

We know what $g_{\rho\sigma}$ is, from (1.19). And, using the relationship (1.21), we can compute all of the derivatives $dx^\rho/dx^{\mu'}$. So, from here the problem is straightforward plug-and-chug.

As a warm-up example, let's compute $g_{t't'}$. Both t and ϕ are functions of t', while r and z are not. And, $g_{tt} = -c^2$, while $g_{\phi\phi} = r^2$. So,

$$\begin{aligned} g_{t't'} &= \frac{\partial x^\rho}{dt'} \frac{\partial x^\sigma}{dt'} g_{\rho\sigma} = \frac{\partial t}{\partial t'} \frac{\partial t}{\partial t'} g_{tt} + \frac{\partial \phi}{\partial t'} \frac{\partial \phi}{\partial t'} g_{\phi\phi} \\ &= (1)^2(-c^2) + (\omega)^2(r^2) = -c^2 + \omega^2 r^2 \ . \end{aligned} \tag{1.23}$$

The last step is to express r in terms of the new coordinates. We find simply that

$$g_{t't'} = -c^2 + \omega^2(x'^2 + y'^2) \ . \tag{1.24}$$

A more difficult example is computing $g_{x'x'}$. r and ϕ are both functions of x', and so

$$\begin{aligned} g_{x'x'} &= \frac{\partial x^\rho}{dx'} \frac{\partial x^\sigma}{dx'} g_{\rho\sigma} = \frac{\partial r}{\partial x'} \frac{\partial r}{\partial x'} g_{rr} + \frac{\partial \phi}{\partial x'} \frac{\partial \phi}{\partial x'} g_{\phi\phi} \\ &= \left(\frac{x'}{\sqrt{x'^2 + y'^2}} \right)^2 (1) + \left(-\frac{y'}{x'^2 + y'^2} \right)^2 (r^2) \\ &= \frac{x'^2}{x'^2 + y'^2} + \frac{y'^2}{x'^2 + y'^2} = 1 \ . \end{aligned} \tag{1.25}$$

Note that in the last line we used $r^2 = x'^2 + y'^2$ to express everything in terms of the new coordinates, and we end up with everything simplifying such that $g_{x'x'} = 1$. Similarly, you should also find that $g_{y'y'} = 1$ (the computation is basically the same). And, $g_{z'z'}$ is trivial, since only $z = z'$ depends on z':

$$g_{z'z'} = \left(\frac{\partial z}{\partial z'} \right)^2 g_{zz} = 1 \ . \tag{1.26}$$

To summarize, the diagonal elements of the new metric are

$$g_{t't'} = -c^2 + \omega^2(x'^2 + y'^2) \ , \quad g_{x'x'} = g_{y'y'} = g_{z'z'} = 1 \ . \tag{1.27}$$

We're not yet done, though. There can also be off-diagonal elements $g_{t'x'}$, $g_{t'y'}$, $g_{t'z'}$, and so on. Let's first tackle $g_{t'x'}$. For this, we notice that only ϕ depends on both t' and x'. Thus we find

$$\begin{aligned} g_{t'x'} &= \frac{\partial x^\rho}{\partial t'} \frac{\partial x^\sigma}{\partial x'} g_{\rho\sigma} = \frac{\partial \phi}{\partial t'} \frac{\partial \phi}{\partial x'} g_{\phi\phi} \\ &= (\omega) \left(-\frac{y'}{x'^2 + y'^2} \right) (r^2) = -\omega y' \ . \end{aligned} \tag{1.28}$$

Once again, in the last equality we expressed $r^2 = x'^2 + y'^2$ to make sure that our metric is written only in terms of the new coordinates. A very similar computation (which you should of course verify) shows that $g_{t'y'} = \omega x'$. Then, since the metric is symmetric, we know that $g_{x't'} = g_{t'x'} = -\omega y'$ and $g_{y't'} = g_{t'y'} = \omega x'$.

The next non-trivial calculation to do is $g_{x'y'}$. Both r and ϕ depend on both x' and y', and so

$$\begin{aligned} g_{x'y'} &= \frac{\partial x^\rho}{\partial x'}\frac{\partial x^\sigma}{\partial y'}g_{\rho\sigma} = \frac{\partial r}{\partial x'}\frac{\partial r}{\partial y'}g_{rr} + \frac{\partial \phi}{\partial x'}\frac{\partial \phi}{\partial y'}g_{\phi\phi} \\ &= \left(\frac{x'}{\sqrt{x'^2+y'^2}}\right)\left(\frac{y'}{\sqrt{x'^2+y'^2}}\right)(1) + \left(-\frac{y'}{x'^2+y'^2}\right)\left(\frac{x'}{x'^2+y'^2}\right)(r^2) \\ &= \frac{x'y'}{x'^2+y'^2} - \frac{y'x'}{x'^2+y'^2} = 0 \ . \end{aligned} \tag{1.29}$$

So, everything cancels, and we find $g_{x'y'} = g_{y'x'} = 0$

Lastly, you should convince yourself that $g_{t'z'} = g_{x'z'} = g_{y'z'} = 0$. This is because none of the cylindrical coordinates depend on both z' and t', or z' and x', or z' and y'. Therefore, we find that:

$$\begin{aligned} g_{t't'} &= -c^2 + \omega^2(x'^2+y'^2) \ , \\ g_{x'x'} &= g_{y'y'} = g_{z'z'} = 1 \ , \\ g_{t'x'} &= g_{x't'} = -\omega y' \ , \\ g_{t'y'} &= g_{y't'} = \omega x' \ , \\ g_{x'y'} &= g_{y'x'} = 0 \ , \\ g_{t'z'} &= g_{z't'} = g_{x'z'} = g_{z'x'} = g_{y'z'} = g_{z'y'} = 0 \ . \end{aligned} \tag{1.30}$$

A slightly more compact way to write this is

$$ds^2 = (-c^2 + \omega^2(x'^2+y'^2))dt'^2 + dx'^2 + dy'^2 + dz'^2 - 2\omega y' dt' dx' + 2\omega x' dt' dy' \ . \tag{1.31}$$

We could also write it in matrix form:

$$g_{\mu\nu} = \begin{pmatrix} -c^2+\omega^2(x'^2+y'^2) & -\omega y' & \omega x' & 0 \\ -\omega y' & 1 & 0 & 0 \\ \omega x' & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ . \tag{1.32}$$

All of these are equivalent statements. The advantage of the matrix form is that inverting the metric is easy, and so the inverse metric $g^{\mu\nu}$ can be written as

$$g^{\mu\nu} = \begin{pmatrix} -1/c^2 & -\omega y'/c^2 & \omega x'/c^2 & 0 \\ -\omega y'/c^2 & 1-\frac{\omega^2 y'^2}{c^2} & \omega^2 x' y'/c^2 & 0 \\ \omega x'/c^2 & \omega^2 x' y'/c^2 & 1-\frac{\omega^2 x'^2}{c^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ . \tag{1.33}$$

Unfortunately, when the metric is not diagonal there aren't a lot of shortcuts for computing the inverse metric. You typically just have to write it in matrix form and then use whatever method you like best to invert it.

2 Induced Metrics

5.) The hypersurface, which we will denote by Σ, has $t = T$ and $x^2 + y^2 = R^2$, where T and R are constants. Fixing t to be constant is easy, but fixing $x^2 + y^2$ to be constant is a little tricky to work with in the normal Cartesian coordinates, since it means that x and y depend on one another. We can avoid this difficulty by going to polar coordinates (t, r, ϕ, z), where $r = \sqrt{x^2 + y^2}$ and $\tan\phi = y/x$. The metric in polar coordinates is, as proven on last week's problem set,

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\phi^2 + dz^2 \ . \tag{2.1}$$

In this coordinate frame, the equation governing the surface of interest is simply

$$t = T \ , \quad r = R \ . \tag{2.2}$$

That is, both t and r are held constant. Since they are constant, there is no infinitesimal change dt or dr between two points on the embedded surface Σ. So, we simply need to set $dt = dr = 0$, as well as fix $r = R$. The polar coordinate metric (2.1) thus becomes the following when restricted to Σ:

$$ds^2|_\Sigma = R^2 d\phi^2 + dz^2 \ . \tag{2.3}$$

Now, what is Σ? Well, since t and r are both fixed, this means that the only way to move across the surface are to change the angle ϕ or to move up and down in the z direction. In other words, our hypersurface should correspond to an infinite cylinder, stretching along the z-axis, of a fixed radius R. And, in fact, the induced metric (2.3) confirms this intuition. (2.3) tells us that the distance between two points has two contributions: an arc length contribution that comes from varying ϕ, and a simply length contribution coming from varying z. This is precisely the metric of a cylinder, and so Σ is in fact a cylinder embedded into a four-dimensional Minkowski spacetime.

6.) There are two coordinates that can vary along the hypersurface Σ in this problem, θ and ϕ, while T, a, and b are all constants. The chain rule can therefore be used to express the differentials dt, dx, dy, and dz in terms of $d\theta$ and $d\phi$ as follows:

$$\begin{aligned} dt &= \frac{\partial t}{\partial\theta} d\theta + \frac{\partial t}{\partial\phi} d\phi = 0 \ , \\ dx &= \frac{\partial x}{\partial\theta} d\theta + \frac{\partial x}{\partial\phi} d\phi = -b\sin\theta\, \cos\phi\, d\theta - (a + b\cos\theta)\sin\phi\, d\phi \ , \\ dy &= \frac{\partial y}{\partial\theta} d\theta + \frac{\partial y}{\partial\phi} d\phi = -b\sin\theta\, \sin\phi\, d\theta + (a + b\cos\theta)\cos\phi\, d\phi \ , \\ dz &= \frac{\partial z}{\partial\theta} d\theta + \frac{\partial z}{\partial\phi} d\phi = b\cos\theta\, d\theta \ . \end{aligned} \tag{2.4}$$

From here, we simply plug these relations into the Minkowski metric:

$$\begin{aligned} ds^2|_\Sigma &= -c^2 dt^2 + dx^2 + dy^2 + dz^2 \\ &= -c^2(0) + (-b\sin\theta\, \cos\phi\, d\theta - (a + b\cos\theta)\sin\phi\, d\phi)^2 \\ &\quad + (-b\sin\theta\, \sin\phi\, d\theta + (a + b\cos\theta)\cos\phi\, d\phi)^2 + (b\cos\theta\, d\theta)^2 \ . \end{aligned} \tag{2.5}$$

Notice that each term that is squared above will yield things that look like $d\theta^2$, $d\phi^2$, and $d\theta d\phi$. In order to simplify this as much as possible, we will group the terms that multiply $d\theta^2$, $d\phi^2$, and $d\theta d\phi$ separately:

$$\begin{aligned} ds^2|_\Sigma = &\left(b^2\sin^2\theta\cos^2\phi + b^2\sin^2\theta\sin^2\phi + b^2\cos^2\theta\right) d\theta^2 \\ &+ \left((a + b\cos\theta)^2\sin^2\phi + (a + b\cos\theta)^2\cos^2\phi\right) d\phi^2 \\ &+ (2b(a + b\cos\theta)\sin\theta\sin\phi\cos\phi - 2b(a + b\cos\theta)\sin\theta\sin\phi\cos\phi)\, d\theta d\phi \ . \end{aligned} \tag{2.6}$$

Notice that the cross-terms that contribute to the $d\theta d\phi$ term cancel; this is exactly analogous to how such cross-terms also cancel in the polar and spherical coordinate metrics we looked at last week. Moreover, there are lots of terms that simplify by using the trig identities $\sin^2\theta + \cos^2\theta = 1$ and $\sin^2\phi + \cos^2\phi = 1$. We end up finding that:

$$\boxed{ds^2|_\Sigma = b^2 d\theta^2 + (a + b\cos\theta)^2\, d\phi^2}\,. \tag{2.7}$$

7.) **Bonus:** Now that we have identified the induced metric (2.7), we want to figure out what this hypersurface Σ actually is, physically. The surface is two-dimensional, spanned by the coordinates θ and ϕ. The first term in (2.7) looks similar to the $r^2 d\phi^2$ term in the polar coordinate metric, and so intuitively this $b^2 d\theta^2$ term should correspond to some kind of circle of radius b. The second term is a little stranger, though, because the thing multiplying $d\phi^2$ is $(a + b\cos\theta)^2$, which is not just a simple circle. So, the arc length along the θ-direction only depends on the angle θ, while the arc length along the ϕ direction depends on both ϕ and θ.

What kind of shape can accomodate two independent angles, with two separate arc lengths that satisfy the properties above? The answer is a two-dimensional torus, usually denoted T^2, and depicted in figure 1. Or, in simpler terms, a donut. The parameter a in our metric is equivalent to the parameter R_1 in the figure, i.e. the major axis radius, and the parameter b is equivalent to the parameter R_2 in the figure, i.e. the minor axis radius. The coordinate ϕ is equivalent to u in the figure, and is the angle in the $x - y$ plane, while the coordinate θ is equivalent to the parameter v in the figure, and it is the angle along the torus as measured from a ring in the torus.

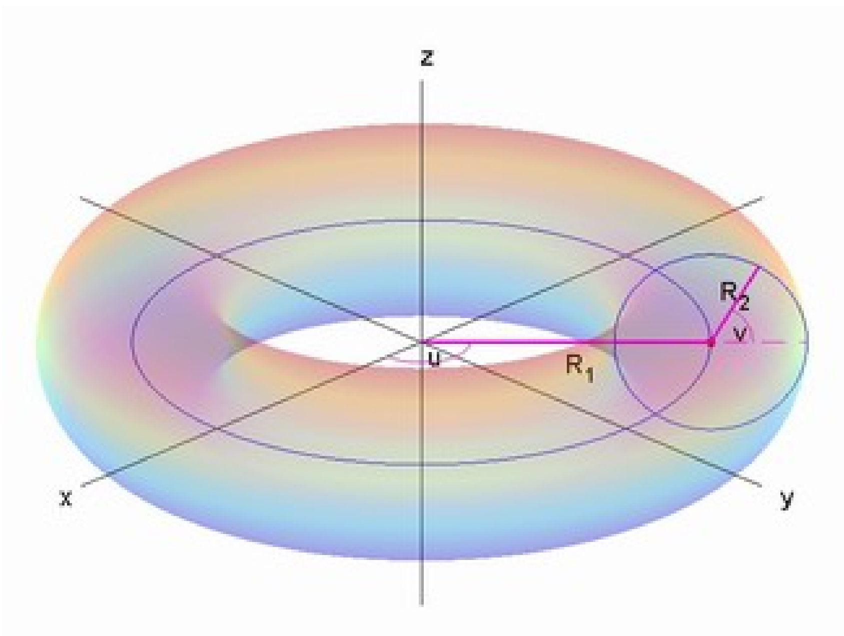

Figure 1. A torus, as embedded into the Cartesian space $\mathbb{R}^3$. I am not a good artist so I stole this picture from the internet. To match with the conventions in our problem, you have to set $a = R_1$, $b = R_2$, $\phi = u$, and $\theta = v$.

3 Spheres and Loathing in Las Vegas

8.) See question 12 on problem set 3 for a detailed answer on how to derive the sphere metric. The result is that if we define spherical coordinates by

$$x = r\sin\theta\,\sin\phi\ ,\quad y = r\sin\theta\,\cos\phi\ ,\quad z = r\cos\theta\ , \tag{3.1}$$

then the metric in these coordinates is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2\ . \tag{3.2}$$

Equivalently, we can say that the only non-zero components of the metric are $g_{rr} = 1$, $g_{\theta\theta} = r^2$, and $g_{\phi\phi} = r^2\sin^2\theta$.

9.) The metric is diagonal, and so the inverse is easy to compute; it simply amounts to letting $g^{rr} = 1/g_{rr}$, $g^{\theta\theta} = 1/g_{\theta\theta}$, and $g^{\phi\phi} = 1/g_{\phi\phi}$. Combining this with our answer to the previous part, we find that

$$g^{rr} = 1\ ,\quad g^{\theta\theta} = \frac{1}{r^2}\ ,\quad g^{\phi\phi} = \frac{1}{r^2\sin^2\theta}\ , \tag{3.3}$$

and all other components of the inverse metric are zero. Make sure you understand that this yields $g^{\mu\nu}g_{\nu\rho} = \delta^\mu_\rho$!

10.) We can take (3.1) and solve for the spherical coordinates (r, θ, ϕ) in terms of the Cartesian coordinates. The result of this is

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2}\ , \\ \theta &= \arccos\left(\frac{z}{r}\right) = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\ , \\ \phi &= \arctan\left(\frac{y}{x}\right)\ . \end{aligned} \tag{3.4}$$

Then, to find the path in spherical coordinates we simply need to plug in for $x(\lambda)$, $y(\lambda)$, and $z(\lambda)$, and then simplify as much as possible. The result is

$$r(\lambda) = \sqrt{1+\lambda^2}\ ,\quad \theta(\lambda) = \arccos\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)\ ,\quad \phi = \lambda\ . \tag{3.5}$$

Now, we need to compute tangent vectors in both coordinate frames. When we parameterize a path by some parameter λ (usually referred to as the "affine parameter"), the tangent vector is defined by

$$V^\mu = \frac{dx^\mu}{d\lambda}\ . \tag{3.6}$$

So, we simply need to differentiate the coordinates (x, y, z) as well as the coordinates (r, θ, ϕ) with respect to λ. The respective tangent vectors are:

$$\begin{aligned} &\text{Cartesian coordinates:} \quad && V^\mu = (-\sin\lambda, \cos\lambda, 1) \\ &\text{Spherical coordinates:} \quad && V^\mu = \left(\frac{\lambda}{\sqrt{1+\lambda^2}}, -\frac{1}{1+\lambda^2}, 1\right) \end{aligned} \tag{3.7}$$

These are clearly not the same, which makes sense; vectors have indices and thus transform under coordinate transformations.

11.) If we hold $r = R$ fixed, then there is no sense in which we can talk about infinitesimal changes dr in the radial direction; r is constant. So, the line element ds^2 will now no longer have any dr terms in it, we simply set them to zero. Additionally, we fix $r = R$ everywhere it appears in ds^2. The result is

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta \, d\phi^2 \ . \tag{3.8}$$

This describes the metric of a two-dimensional surface, the two-sphere S^2 of radius R. We can equivalently write this as

$$g_{\theta\theta} = R^2 \ , \quad g_{\phi\phi} = R^2 \sin^2\theta \ . \tag{3.9}$$

The inverse metric is then simply

$$g^{\theta\theta} = \frac{1}{R^2} \ , \quad g^{\phi\phi} = \frac{1}{R^2 \sin^2\theta} \ . \tag{3.10}$$

12.) At any fixed angle θ, the height $z = r\cos\theta$ along the sphere is fixed. However, the angle ϕ along the $x - y$ plane is not fixed, and so we can still go all the way around the circle at a fixed height z. The resulting curve is still a circle, although the radius of this circle is less than (or equal to) the sphere radius R, depending on what value we fix θ to. So, by fixing θ and letting ϕ be free, we define a circle (i.e. an S^1) embedded within our S^2. Since θ is fixed, $d\theta = 0$, and so the induced metric is

$$ds^2 = R^2 \sin^2\theta \, d\phi^2 \equiv \tilde{R}^2 d\phi^2 \ , \tag{3.11}$$

where we have defined $\tilde{R} \equiv R\sin\theta$. That is, the induced metric on the S^1 is that of a circle of radius $\tilde{R}$. When $\theta = 0$ or $\theta = \pi$, we are at the north or south pole, and the size of the S^1 goes to zero, $\tilde{R} = 0$. When we are at $\theta = \pi/2$, we are on the equator of the sphere, and the radius of the S^1 is $\tilde{R} = R$.

13.) **Bonus:** the n-sphere S^n is defined as the n-dimensional surface of points that are a fixed distance R from the origin. The canonical way to define a metric on S^n is to first label the coordinates on the sphere by $\theta_1, \theta_2, \ldots, \theta_{n-1}, \theta_n$, where the angles range over

$$\theta_{1,\ldots,n-1} \in [0, \pi) \ , \quad \theta_n \in [0, 2\pi) \ . \tag{3.12}$$

In terms of these coordinates, the metric for an n-sphere of radius R is given by

$$\begin{aligned} ds^2 = & \ R^2 d\theta_1^2 \\ & + R^2 \sin^2\theta_1 \, d\theta_2^2 \\ & + R^2 \sin^2\theta_1 \sin^2\theta_2 \, d\theta_3^2 \\ & + \ldots \\ & + R^2 \sin^2\theta_1 \sin^2\theta_2 \ \ldots \sin^2\theta_{n-1} d\theta_n^2 \ . \end{aligned} \tag{3.13}$$

Now that we've established what an n-sphere is, the goal of this question is to prove that the n-sphere S^n is embeddable into the Cartesian space $\mathbb{R}^{n+1}$.

The easiest way to prove this is to explicitly construct the embedding and show that the induced metric on the S^n is precisely the metric we expect, namely the one given above in (3.13). Our starting point is the usual Cartesian metric on $\mathbb{R}^{n+1}$, given by

$$ds^2 = dx_1^2 + dx_2^2 + \ldots + dx_n^2 \ , \tag{3.14}$$

where the coordinates $x_1, \dots, x_{n+1}$ all range over the whole real line. From here, we can establish a set of spherical coordinates $(r, \theta_1, \dots, \theta_n)$, defined in terms of the original Cartesian coordinates as follows:

$$\begin{aligned}
x_1 &= r\cos\theta_1 \ , \\
x_2 &= r\sin\theta_1\cos\theta_2 \ , \\
x_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3 \ , \\
&\ \vdots \\
x_n &= r\sin\theta_1\dots\sin\theta_{n-1}\cos\theta_n \ , \\
x_{n+1} &= r\sin\theta_1\dots\sin\theta_{n-1}\sin\theta_n \ .
\end{aligned} \tag{3.15}$$

We can use the chain rule to then establish the following:

$$\begin{aligned}
dx_1 &= \cos\theta_1\, dr - r\sin\theta_1\, d\theta_1 \ , \\
dx_2 &= \sin\theta_1\cos\theta_2\, dr + r\cos\theta_1\cos\theta_2\, d\theta_1 - r\sin\theta_1\sin\theta_2\, d\theta_2 \ , \\
dx_3 &= \sin\theta_1\sin\theta_2\cos\theta_3\, dr + r\cos\theta_1\sin\theta_2\cos\theta_3\, d\theta_1 \\
&\quad + r\sin\theta_1\cos\theta_2\cos\theta_3\, d\theta_2 - r\sin\theta_1\sin\theta_2\sin\theta_3\, d\theta_3 \ , \\
&\ \vdots \\
dx_n &= \sin\theta_1\dots\sin\theta_{n-1}\cos\theta_n\, dr \\
&\quad + r\cos\theta_1\sin\theta_2\dots\sin\theta_{n-1}\cos\theta_n\, d\theta_1 \\
&\quad + r\sin\theta_1\cos\theta_2\sin\theta_3\dots\sin\theta_{n-1}\cos\theta_n\, d\theta_2 \\
&\quad + \dots \\
&\quad + r\sin\theta_1\dots\sin\theta_{n-2}\cos\theta_{n-1}\cos\theta_n\, d\theta_{n-1} \\
&\quad - r\sin\theta_1\dots\sin\theta_{n-1}\sin\theta_n\, d\theta_n \ , \\
dx_{n+1} &= \sin\theta_1\dots\sin\theta_{n-1}\sin\theta_n\, dr \\
&\quad + r\cos\theta_1\sin\theta_2\dots\sin\theta_{n-1}\sin\theta_n\, d\theta_1 \\
&\quad + r\sin\theta_1\cos\theta_2\sin\theta_3\dots\sin\theta_{n-1}\sin\theta_n\, d\theta_2 \\
&\quad + \dots \\
&\quad + r\sin\theta_1\dots\sin\theta_{n-2}\cos\theta_{n-1}\sin\theta_n\, d\theta_{n-1} \\
&\quad + r\sin\theta_1\dots\sin\theta_{n-1}\cos\theta_n\, d\theta_n \ .
\end{aligned} \tag{3.16}$$

The computation from here is somewhat tedious, but we now need to plug these differentials into the $\mathbb{R}^{n+1}$ metric (3.14). I won't repeat the computation here, since it's a little long, but you should be able to see that all off-diagonal terms cancel, and all diagonal terms simplify via the usual trig identities. The result is that

$$\begin{aligned}
ds^2 = dr^2 &\\
&+ r^2 d\theta_1^2 \\
&+ r^2\sin^2\theta_1\, d\theta_2^2 \\
&+ r^2\sin^2\theta_1\sin^2\theta_2\, d\theta_3^2 \\
&+ \dots \\
&+ r^2\sin^2\theta_1\sin^2\theta_2\ \dots\sin^2\theta_{n-1} d\theta_n^2 \ .
\end{aligned} \tag{3.17}$$

In this coordinate frame, it should be clear that by fixing $r = R$, the resulting hypersurface should be an n-sphere with radius R. In particular, when we fix $r = R$ and set $dr = 0$, the resulting induced metric matches the S^n metric (3.13) exactly. Thus, we have shown concretely that there is an embedding of S^n into $\mathbb{R}^{n+1}$ such that the induced metric on

S^n is exactly the well-known metric on S^n.

In slightly more mathematical terms, we can define a mapping f from S^n to $\mathbb{R}^{n+1}$ by $f(\theta_1, \ldots, \theta_n) = (x_1, \ldots, x_n)$, where the x_i are related to the θ_i by the coordinate transformation (3.15) with $r = R$ fixed to some constant. This mapping is injective, essentially because $\cos\theta_i$ is single-valued on the interval $\theta_i \in [0, \pi)$, which guarantees that no two points on S^n map to the same point in $\mathbb{R}^{n+1}$ when acting with f. Moreover, f is a structure-preserving map, in the sense that the metric of S^n is preserved under this mapping, as we have just established. We therefore conclude that

$$f : S^n \hookrightarrow \mathbb{R}^{n+1} \,, \tag{3.18}$$

i.e. that this mapping f actually defines a proper embedding of S^n into $\mathbb{R}^{n+1}$.